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Two-dimensional avalanches as stochastic Markov processes

E. Morales-Gamboa
Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, 04510 México D.F., Mexico

J. Lomnitz-Adler and V. Romero-Rochín
Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 México, D.F., Mexico

R. Chicharro-Serra and R. Peralta-Fabi
Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, 0410 México D.F., Mexico
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A theory is proposed to describe two-dimensional avalanches in granular materials. The theoretical framework is based on a set of experiments in which avalanches are generated intermittently by means of a rotating drum. From measurements of the instantaneous average angle of the free surface, $\alpha(t)$, the relevant quantities that characterize avalanches, such as critical angles and times, are found to be given by distributions. It is then shown that $\alpha(t)$ is a stochastic Markov process. Consequently, a master equation is constructed for $P(\alpha,t)$, the conditional probability that the slope is α, at time t, given an initial state. Finally, a consistency test is carried out, finding full agreement with the experimental data.

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The history of a steady state punctuated by abrupt catastrophic events is the generic feature that excites interest in the study of avalanches. These can be viewed as prototypical of a class of extended nonlinear systems [1–3] for which a variety of different theoretical treatments has been developed to describe their evolution [3–14]; most of these approaches are in some sense microscopic descriptions, since they follow the motion of individual grains. A macroscopic theory to describe some of the coarse features, leading eventually to a formulation of the appropriate constitutive relations for granular materials, is still the subject of considerable research [11–14]. A successful application of an approach similar to the one presented here is on long-term earthquake prediction [15].

In this Rapid Communication we present a general theoretical approach to describing the evolution of two-dimensional (2D) avalanches as a stochastic process. The theory stems from experiments that focus on the instantaneous (average) angle of the free surface of the system, $\alpha(t)$, without considering the microscopic degrees of freedom explicitly. The observations lead in a natural way to a description in terms of distributions and probability densities. Further analysis shows that avalanches are such that $\alpha(t)$ is a stochastic Markov process, and as a consequence the evolution of the system can be described by a master equation [16]. One of the virtues of this level of description is that it provides a solid basis for a fully macroscopic theory, while incorporating the sensitivity to microscopic configurations characteristic of granular media.

Experiments and simulations. The experimental setup consists of a horizontal rotating acrylic cylinder, which is half-filled with granular material, similar to that used by other authors [14,17–19]. We used spherical glass beads of even size with diameter d (3 mm) and a density of 2.34 g/cm^3. The length of the cylinder is such that no more than one layer of beads can be placed between the two faces of the cylinder. With this arrangement, the system can be considered as two dimensional, since all flows take place on a vertical plane. Although the effect of the walls is hard to quantify, it is systematic throughout the experiments; moreover, there are studies that indicate that the lateral walls play a minor role [20]. The system is small in the sense that the diameter ratio $s=D/d$, where D is the cylinder's diameter, is varied between 50 and 150.

The cylinder was rotated with a constant and low angular velocity (Ω) in order to have a clear separation between the average time between avalanches $\langle \delta t \rangle$ and the average duration of an avalanche $\langle \delta\tau \rangle$; typically, the time scales involved were 10 min for $1/\Omega$, 20 sec for δt, and 1 sec for $\delta\tau$. This way the system was far from the "fluidization" regime [14,21]. All the experimental data that are reported here correspond to the case $s=50$ and $\Omega=0.6°/\text{sec}$.

The front face of the rotating cylinder was continually registered with a video camera. Automatically, and in real time, a frame was taken, digitized, and an average angle was determined by a least-squares fit, using the pixels that constitute the free surface; each value of α took 0.5 sec. In this way we were able to measure $\alpha(t)$ with a 1.5% error. Figure 1 shows part of the time series of 3338 avalanches analyzed here.

As can be seen from Fig. 1, after the termination of an avalanche there is a "loading" time interval δt, during which $\alpha(t)$ increases steadily with Ω, until the system reaches an (instability) angle ϕ, from which the next avalanche of short-duration occurs.

From $\alpha(t)$ we extracted the following information: $\xi(\phi)$, the distribution of angles ϕ at which avalanches ini-

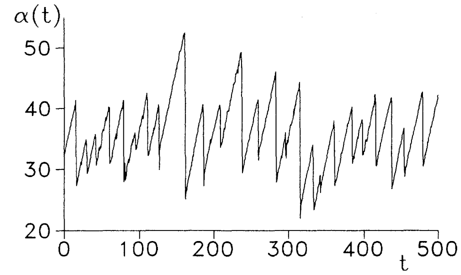

FIG. 1. Typical experimental output showing the average (slope) angle of the free surface $\alpha(t)$ (deg) as a function of time (sec).

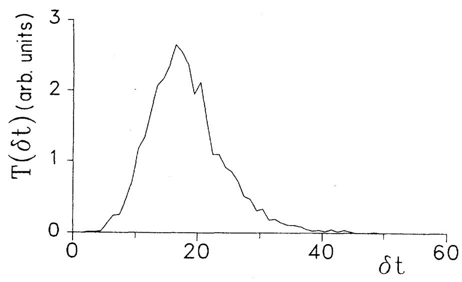

FIG. 3. Distribution of times between avalanches, $T(\delta t)$.

tiate; $\chi(\vartheta)$, the distribution of angles ϑ at which avalanches terminate; and $T(\delta t)$, the distribution of time intervals between the end of an avalanche and the beginning of the next one. Since the average duration of an avalanche $\langle \delta\tau \rangle$ is of the order of 1 sec, to determine the distribution of time duration of avalanches requires an additional procedure. However, for the present purposes and based on the different time scales involved, avalanches can be treated as instantaneous (see Fig. 1).

The distributions of initial and final angles $\xi(\phi)$ and $\chi(\vartheta)$ are shown in Fig. 2. The distribution of times between avalanches $T(\delta t)$ is shown in Fig. 3. We emphasize the fact that initial and final angles, as well as the time between avalanches, vary over a considerable range [14].

The apparent lack of regularity in the "instantaneous" angle of the system suggests that we are facing a problem whose main features are better described in terms of a stochastic process rather than in terms of a deterministic one. It is then necessary to find out whether there is a correlation between the loading process and the avalanche itself, and if there are any memory effects between consecutive avalanches [14]. To resolve these issues, we compared the statistical properties of the true time series with a numerical simulation of the series, which is Markovian by construction.

Three different numerical simulations were conducted in the following way. In the first simulation we generated values for ϑ and δt, at random and independently, from the experimental distributions $\chi(\vartheta)$ and $T(\delta t)$. Then, the angle at which the next avalanche is initiated was calculated using

$$\phi=\vartheta=\Omega\delta t \ . \tag{1}$$

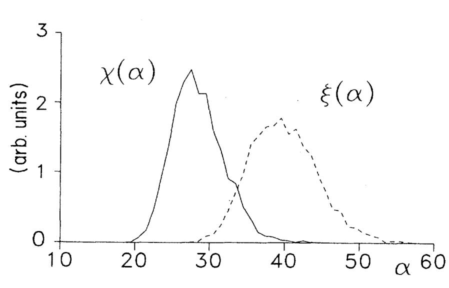

FIG. 2. Angle distributions for avalanche initiation, $\xi(\phi)$ (— — —), and avalanche termination, $\chi(\vartheta)$ (——).

By iterating this procedure we obtained a time series $\alpha_c(t)$. A comparison was made between different quantities and statistical properties of both sets of data (simulated and experimental), such as the power spectrum, the pair distribution function, and the distribution of initiation angles. In particular, in Fig. 4 we show the comparison of the results for the $\xi(\phi)$ distributions. The agreement between the results from the simulation and the experimental data supports both a Markovian description and the independence between the loading and the discharge processes.

In the other two simulations, either the pairs $\xi(\phi)$ and $T(\delta t)$ or $\xi(\phi)$ and $\chi(\vartheta)$ were prescribed to generate a time series $\alpha_c(t)$. A poor agreement was found between the statistical properties of the experimental $\alpha(t)$ and the numerical time series $\alpha_c(t)$. This can be understood as follows. In the ξ-T case the simulation must be made by running time "backwards"; having generated at random the values for ϕ and δt, Eq. (1) is used to determine ϑ, the angle at which the loading process began. The disagreement is a consequence of the strongly dissipative nature of the system; this irreversible behavior must be an essential feature of the theory. For the ξ-χ simulation, although it is not obvious at first sight, the problems occur because once the angles ϑ and ϕ are given it is not necessarily true that a single loading or a single avalanche has taken place.

The successful simulation is the only one that follows the natural time sequence; moreover, it indicates that the mechanisms for initiation and propagation of avalanches are different [14,22]. That is, the loading process is a static one in which friction forces dominate, while the

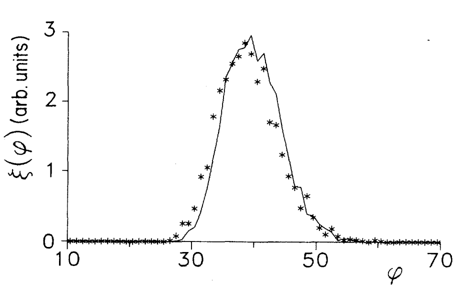

FIG. 4. Comparison between the simulated (* * *) and the experimental (——) angle distributions of avalanche initiation $\xi(\phi)$. The correlation coefficient is 0.9805.

avalanche proper is a fully dynamic stage where collisional processes are determinant. This should be an important point in understanding avalanche dynamics.

We also determined $P(\alpha,t;\alpha_0)\equiv P(\alpha,t)$, the normalized probability distribution that the system takes the angle α at time t, given that initially the angle was α_0; and $P_s(\alpha)$, the stationary, or long-time limit, of $P(\alpha,t)$. The sequence in Fig. 5 illustrates the fast time approach to the stationary value. In Fig. 5(a), $P(\alpha,t)$ is determined after a time equal to half the average time between avalanches ($\langle\delta t\rangle/2$); the large peak corresponds to those cases in which an avalanche has not yet taken place, and the angle has grown steadily with Ωt. In Fig. 5(b), the time is $2\langle\delta t\rangle$; in the majority of the cases at least one avalanche has taken place and in very few, one is yet to happen. Figure 5(c) is the "infinite time" probability distribution; $P(\alpha,t)$ remains unchanged after $5\langle\delta t\rangle$, as the system has "forgotten" the initial state. In what follows we shall be concerned essentially with the stationary case.

Theoretical treatment. Supported by the experiments, we assume that $\alpha(t)$ constitutes a stochastic Markovian process. A direct consequence is that $P(\alpha,t)$ satisfies the Chapman-Kolmogorov equation. An alternative and equivalent formulation is given by the following master equation [16]:

$$\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\alpha}\right]P(\alpha,t)=\int_0^{2\pi}[W(\alpha,\alpha')P(\alpha',t)-W(\alpha',\alpha)P(\alpha,t)]d\alpha' . \quad (2)$$

$W(\alpha,\alpha')$ is the transition probability per unit time that the system whose angle is α' evolves by way of an avalanche to the final angle α. The presence of $\Omega\partial/\partial\alpha$ in the streaming operator stems from the fact that the angle increases steadily in time due to the rotation of the system. We define $Q_0(\alpha)$ as the probability per unit time that a transition (avalanche) begins at an angle α:

$$Q_0(\alpha)=\int_0^{2\pi}W(\alpha',\alpha)d\alpha' . \quad (3)$$

The next step, a crucial one, is to verify that the system is correctly described by the master equation, Eq. (2). In view of the fast approach to the stationary distribution (Fig. 5), we concentrate on the latter.

To proceed we must give a prescription for $W(\alpha,\alpha')$. To do so, we note that together with showing the Markovian nature of the avalanche processes, we also showed that the loading and the discharge mechanisms are independent. Consequently, we can write

$$W(\alpha,\alpha')=\chi(\alpha)Q_0(\alpha')+C(\alpha,\alpha') , \quad (4)$$

where $C(\alpha,\alpha')$ stands for the correction implied by the evident fact that avalanches must terminate at a lower angle than the one at which they are initiated ($\alpha'>\alpha$). As an approximation, we neglect the corresponding correlation so that the kernel $W(\alpha,\alpha')$ becomes completely factorable. Incidentally, this correlation is reflected by the small overlap between the distributions $\xi(\alpha)$ and $\chi(\alpha)$ (Fig. 2).

Also, we note that we have experimental access to the normalized distributions of initiation angles $\xi(\alpha)$ and final or termination angles $\chi(\alpha)$. The former is equal to the probability that the system reaches an angle α, $P_s(\alpha)$, times the probability per unit time that an avalanche is initiated with this value, $Q_0(\alpha)$; that is,

$$\xi(\alpha)=c_1^{-1}P_s(\alpha)Q_0(\alpha) , \quad (5)$$

where c_1 is a normalization constant. Note that c_1 is the average number of avalanches per unit time; in a given experiment, it is precisely the total number of avalanches divided by the total observation time ($t_{\rm obs}\gg\delta t$).

Using Eq. (2) for the stationary case and substituting Eqs. (3)–(5), we have, after one integration,

$$P_s(\alpha)-P_s(0)=\frac{c_1}{\Omega}\int_0^{\alpha}d\alpha'[\chi(\alpha')-\xi(\alpha')] . \quad (6)$$

We now proceed with a consistency test between the proposed theory and the experiments. Since we can measure all the quantities involved in the above equation, with *no adjustable parameters,* we can numerically evaluate both sides. The comparison is shown in Fig. 6. The agreement justifies the approach we have followed, expressed in Eq. (2), at least in the stationary case, as well as the approximation introduced upon factoring the kernel.

Remarks. The two-dimensional experimental arrangement we have used allows for a systematic and precise procedure to determine the values of the mean angles. Also, it offers the possibility to measure directly the instantaneous local grain densities; we found that there are only scattered domains of local order, separated by disclinations, voids, and random 2D packing regions, as op-

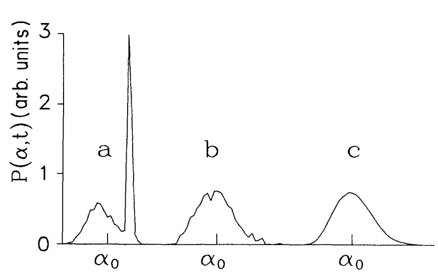

FIG. 5. Time-dependent probability distribution $P(\alpha,t)$, at times $\langle\delta t\rangle/2$, $2\langle\delta t\rangle$, and the long-time limit. The latter is unchanged after $4\langle\delta t\rangle$. Not shown is the initial distribution $P(\alpha,0)=\delta(\alpha-\alpha_0)$; $\alpha_0=34°$.

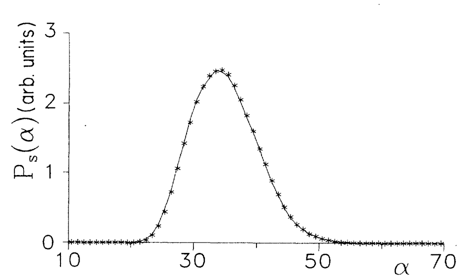

FIG. 6. Experimental test for the master equation, Eq. (6): (——) for left-hand side and (* * *) for right-hand side. The correlation coefficient is 0.9998.

posed to the crystal-like arrangement one might expect. This structure, of short-range order and long-range disorder, is reminiscent of glassy materials [11]. The randomness we have found is in part a reflection of this disorder.

Figure 1 illustrates one particular feature that is seldom emphasized in the literature; namely, that there is considerable dispersion in the angles of initiation and termination of avalanches. Following Coulomb [23], there has been a tendency to assume that there is a unique critical angle that controls both the initiation and the propagation of avalanches. In fact, these systems were expected to have some features of self-organized criticality [4]; that this is not the case has been already noted [24]. At first sight it appears that our analysis is consistent with a second class of observations and models [14,18,19,25,26–28], which presume that such systems have two critical angles: one associated with the initiation of an avalanche (ϕ) and another associated with its termination (ϑ). We believe that the fluctuations in the values of the angles of avalanche initiation and termination are a fundamental property of the system, and must be described in terms of distributions. The nonuniqueness of ϑ and ϕ is a consequence of a strong dependence of the system's evolution on details of the microscopic configuration [13]; this sensitivity is included in the stochastic treatment. This dependence, together with size and boundary effects, is incorporated into the, not yet fully understood, structure of the transition probability kernel $W(\alpha,\alpha')$.

Concerning the size, we found it instructive to deal with rather small systems, since in this case statistical fluctuations tend to be enhanced, providing insight into the nature of the dynamics, which may not always be apparent in a larger system; furthermore, end or size effects can readily be explored [14,19,25,29]. We have repeated the experiment with larger systems ($s=100,150$); while the distributions themselves are slightly modified, being displaced and narrower, none of the principal features of the experiments reported here is modified.

Finally, we speculate that the generalization to three-dimensional inhomogeneous systems will not change the essential points of our analysis. That is, the master equation should remain an appropriate description for a wider class of dry and smooth granular materials.

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